Building Blocks and Cognitive Building Blocks

Playing to Know the World Mathematically

JULIE SARAMA AND DOUGLAS H. CLEMENTS

The authors explore how children's play can support the development of the foundations of mathematics learning and how adults can support children's representation of—and thus the *mathematization* of—their play. The authors review research about the amount and nature of mathematics found in the free play of children. They briefly discuss how children develop different types of play and describe ways adults can support and guide each of these to encourage an understanding of mathematics and to enhance children's mathematical skills. The authors' activities described in this article and the time to prepare it were partially funded by grants from the Institute of Education Sciences (IES) in the U.S. Department of Education and from the National Science Foundation (NSF).

IN THIS ARTICLE, we explore how children's play supports the development of mathematical ideas and skills. We discuss research that suggests how adults can support children's representation of their play and thus its *mathematization*. We begin by observing children to see how much and what kinds of mathematics we can actually find in the free play of children. Next, we briefly review children's development of different types of play and describe ways adults can support and guide each of these in order to encourage children's mathematical development.

Everyday Play and Mathematics

Parents and teachers often notice that children engage in informal mathematical activity during free play. Preschoolers explore patterns and shapes, compare sizes, and count things. But how often do they do this? What does it mean to children's development? Two researchers videotaped ninety children, four- to five-years old, as they played. Some of them came from low-income families, others from middle-income families. The researchers examined the ninety episodes, each fifteen-minutes long. They observed six categories of mathematics content in the children's play activities (Seo and Ginsburg, 2004, all examples are from this team's observations).

Classification

This category includes grouping, sorting, or categorizing by attributes. A child cleaned up the blocks on the rug, for example, by taking one block at a time and placing it in a box that contained the same size and shape of blocks. Also a girl took all the plastic bugs out of the container and sorted them by type of bug and then by color. They were classifying.

Magnitude

Children engaging in activities under this category are describing or comparing the size of objects. Two boys, for example, built structures with LEGO blocks. One said to the other, "Look at mine. Mine is big!" The other protested, "Mine is bigger!" They placed their LEGO structures side by side to compare them. In another instance, when one of the girls in the study brought a newspaper to the art table to cover it, another remarked, "This isn't big enough to cover the table." These boys and girls were considering the mathematical concept of magnitude.

Enumeration

This category includes saying number words, counting, instantly recognizing a number of objects (called *subitizing* in mathematics), or reading or writing numerals. A boy took out all the beads in a box, for example, and put them on a table. He said, "Look! I got one hundred!" He started counting them to check his assertion. Others joined in the counting, and they did count up to one hundred, with few errors. In another case, three girls drew pictures of their families and discussed how many brothers and sisters they had and how old their siblings were. These kids were enumerating.

Dynamics

Those engaged in activities related to this category put things together, take them apart, or explore motions such as flipping. Several girls, for example, flattened a ball of clay into a disk, cut it, and made "pizza," clearly working on the dynamics of their object.

Pattern and Shape

This category includes identifying or creating patterns or shapes or exploring geometric properties. In one example, a child made a bead necklace, creating a yellow-red color pattern. In another, a boy put a double-unit block on the rug, two unit blocks on the double-unit block, and triangular blocks in the middle, building a symmetrical structure. These children were playing with pattern and shape.

Spatial Relations

The final category includes describing or drawing a location or direction. For example, one girl put a dollhouse couch beside a window. Another moved it to the center of the living room, saying, "The couch should be in front of TV." Or a boy asked another where he found the button puzzle he was playing with. "There," said the latter, pointing to a storage unit in the block area. The first boy went to the storage unit and asked him again, "Where?" The second boy replied, "Second one . . . right side, no, the left side," (adapted from Seo and Ginsburg 2004, 93–94).

The range of mathematics in the study was impressive. Even more so was the frequency with which children engaged in math activities. About 88 percent of children engaged in at least one math activity during their play. Overall, the children showed at least one instance of mathematical activity 43 percent of the time they were observed. These actions may have been brief episodes during the minutes of play the study observed, but there is little doubt that children are involved in mathematics for a considerable portion of their free play (Seo and Ginsburg 2004).

Although the level of involvement varied by individual, it was remarkably similar despite family income—44 percent of the children from low-income families, 43 percent of middle-income children, and 40 percent of upper-income youngsters. Further, there were no significant gender differences. A related study did reveal that Chinese children engage in considerably greater amounts of these types of play, particularly in the category of pattern and shape (Ginsburg, Ness, and Seo 2003), to which we will return later.

The frequency with which children engaged in mathematical play was not the same in the different categories of Ginsburg's studies. The greatest frequency was in pattern and shape (21 percent), magnitude (13 percent), enumeration (12 percent), dynamics (5 percent), spatial relations (4 percent), and classification (2 percent). Most adults think the math skills of children are limited to simple verbal counting and shape recognition, but this study reveals a surprisingly rich grasp among the very young of these basic mathematical categories.

Indeed, the children's everyday experiences form an intuitive, implicit conceptual foundation for mathematics. Later, children represent and elaborate on these ideas—creating models of an everyday activity with mathematical objects, such as numbers and shapes; engaging in mathematical actions, such as counting or transforming shapes; and using mathematics to build structures. We call this process *mathematization* (Sarama and Clements, 2009). That is, when children play a game and recognize that they cannot win on the next move because they need a seven and the largest number they can roll is a six, they have represented the game situation with numbers and have used mathematical reasoning. Children who recognize that a floor can be tiled with regular hexagons because "the angles fit together" have modeled an aspect of their world with geometry.

Further, recognizing the difference between *foundational* and *mathematized* experiences is necessary to avoid confusion about the type of activity in which children are engaged (Kronholz 2000). We need to recognize this difference because children need both and, unfortunately, adults often do not provide the mathematics experiences. For example, observations across all settings of a full day in the lives of three-year-olds revealed remarkably few activities, lessons, or episodes of play with mathematical objects—60 percent of the children had no such experience across 180 observations (Tudge and Doucet 2004). Factors such as race-ethnicity, socioeconomic status, and setting (home or child care) did not significantly affect this low frequency. We will return to this issue, but first we discuss the development of different types of play among children.

Development of Different Types of Play

Children engage in different types of play as they develop (Monighan-Nourot 1987; Piaget 1962). *Sensorimotor play* involves learning and repeating action sequences, such as sucking, grasping, clapping, or pouring water. It makes up over 50 percent of all the free activity engaged in by children up to two years of age but declines by about 33 percent before they reach five years of age and another 14 percent or so by age six or seven. However, sensorimotor play remains a part of more sophisticated types of play.

Symbolic or *pretend play* emerges when a child is about fifteen months old, and it develops throughout the preschool years. Because it engenders the growth of representation and decontextualization, symbolic play is important as a child grows for understanding more sophisticated mathematical concepts, up through algebra. As an example of symbolic play, a two-year-old might set a table with toy plates, silverware, and plastic food, copying what he has seen at home. A three-year-old might use a flat piece of wood for a plate and cylinder block for a glass. A four- or five-year-old might imagine the dishes and the roles of family members at dinner, including various interactions and plots related to those interactions.

There are three types of symbolic play: constructive, dramatic, and rule governed. In *constructive play*, children manipulate objects to make something. This constitutes about 40 percent of three-year-olds' play and 50 percent of the play of four- to six-year-olds (Monighan-Nourot et al. 1987). The attraction for the child lies in playing with alternate ways of building something. Many of the examples of free play in the previous section fall into this category, such as the girls' classification of bugs, clay pizzas, and the yellow-red necklace. Clearly, constructive play is well named, as children are also building mathematical ideas and strategies.

Dramatic play involves substituting some imaginary situation for the children's immediate environment. Parten observed that this play may be solitary, parallel, or group play, which Smilansky calls sociodramatic play (Monighan-Nourot et al. 1987). Depending on their ages, personalities, and situations, children play in different ways. On average, most two- to three-year-old children engage in parallel play. They play side-by-side, aware of and observing each other. Although they may not seem to some adults to be playing together, they usually want to be playing near each other. Group play is typical of three- to five-year-old children. Girls moving a couch, for example, involve both constructive and sociodramatic play.

Games with rules involve the gradual acceptance of prearranged, often arbitrary rules. Game play is more structured and organized than sociodramatic play. Children from four to seven years of age learn to participate in such games. Younger children play in an improvisational way, with vague idea of rules. For older children, rules are decided beforehand, and alterations must be agreed upon. Even beyond the more obvious number ideas (on dice, cards, and spinners), such games are a fertile ground for the growth of mathematical reasoning, especially strategic reasoning, autonomy, and independence (Kamii 1985). In what follows, we elaborate on the mathematics that may develop in each type of play.

Sensorimotor, or Manipulative, Play

The desire to manipulate things, to explore the physical aspects of the world, motivates sensorimotor play (Elkonin 1978). Sensorimotor play may seem only distantly related to mathematics, but many sensorimotor activities can provide foundations for or direct experience with mathematical ideas. For example, very young children love to jump up and down, march, and chant. Such activities—a mixture of sensorimotor and symbolic processes—build the kind of action sequences common to the basic mathematical concept of *pattern*. Older preschoolers chant, "Up!" (as they jump), "Down!" (as they crouch low), "Up, down; up, down," creating a connected movement-verbal pattern. Music can help deepen these patterns. Early sensorimotor play involving parent and child can emphasize patterns and foundations of other mathematical content. In a popular Chinese game, Count the Insects, a mother holds her baby's hand with the index finger pointing as she says, "Insects fly, fly, fly, fly," waving the index finger each time. The pointing is coordinated with the rhythmic enunciation of the words, laying the groundwork for the one-to-one correspondence between pointing at objects and saying numbers as in counting. Other early parent-child games promote foundational geometric concepts as well as patterning—for example, in another Chinese mother-baby game, Open-Close, where the mother repeatedly forms the baby's hand into a fist as she says "Close," and opens it for "Open" (Monighan-Nourot et al. 1987).

With toddlers, imitating what children do when they play with blocks, sand, or water, and then carefully adding subtle variations, sometimes invites premathematical explorations. For example, the toddlers might see and attempt new ways at balancing or bridging blocks. Indeed, the benefits of block building are deep and broad. Children increase their math, science, and general reasoning abilities when building with blocks (Kamii, Miyakawa, and Kato 2004). Consider how block building develops. Infants either engage in little systematic organization of objects or show little interest in stacking (Forman 1982; Kamii, Miyakawa, and Kato 2004; Stiles and Stern 2001).

Children begin stacking objects at one year, thus revealing an infant's understanding of the spatial relationship of "on" (Kamii, Miyakawa, and Kato 2004). The "next-to" relation develops at about a year-and-a-half (Stiles-Davis 1988). At two-years old, children place each successive block congruently on or next to the block previously placed (Stiles-Davis 1988). They appear to realize that blocks do not fall when so positioned (Kamii, Miyakawa, and Kato 2004).

At three to four years of age, children regularly build vertical and horizontal components within a building (Stiles and Stern 2001). When asked to build a tall tower, they use long blocks vertically because to their goal of making a stable tower, they have added the making of a stable *tall* tower, first using only one block in this fashion, then several (Kamii, Miyakawa, and Kato 2004). At four years, they can use multiple spatial relations, building in multiple directions and with multiple points of contact among components, and showing flexibility in how they compose and integrate parts of the structure. A small number of children will build a tower with all blocks, for example, by arranging triangular blocks, making these parts combine to make a whole (Kamii, Miyakawa, and Kato 2004). This leads us to symbolic constructive play, since most preschoolers enjoying building something.

Symbolic Constructive Play

Preschoolers engage in rhythmic and musical patterns such as jumping rope while singing or chanting. When guided, they can add more complicated, deliberate patterns, such as "clap, clap, slap; clap, clap slap" to their repertoires. They can talk about these patterns and represent the patterns with words. Kindergartners enjoy making up new motions to fit the same pattern, so "clap, clap, slap" is transformed to "jump, jump, fall down; jump, jump, fall down" and soon symbolized as an *a, a, b; a, a, b* pattern. Kindergartners also can describe such patterns with numbers (two of something, then one of something else), creating the first clear links among patterns, numbers, and algebra.

Children with such experiences will intentionally re-create and discuss patterns in their own artwork. A four-year-old in one of our Building Blocks classrooms loved knowing the rainbow colors (ROYGBIV, for red, orange, yellow, green, blue, indigo, violet) and painted rainbows, flowers, and designs that repeated this sequence several times. (Building Blocks is a National Science Foundation–funded research and development project that includes a full preschool mathematics curriculum based on the notion of mathematizing children's play [Clements and Sarama 2007a.])

Constructive play often involves multiple mathematical concepts. Measurement frequently underlies play in water or on a sand table. Kathy Richardson tells of visiting two classrooms in the same day, observing water play in both. Children were pouring in both, but in one they were also excitedly filling different containers with the same cup, counting how many cupfuls they could fit into each container. The only difference between the two classes was that in the latter, the teacher had asked, "I wonder which of these holds the most cupfuls of water?" (Richardson 2004).

Materials such as sand and Play-doh offer many rich opportunities to promote mathematical thinking and reasoning. Adults might provide suggestive materials (cookie cutters), engage in parallel play with children, and raise comments or questions regarding shapes and amounts of things. For example, children might make multiple copies of the one cookie-cutter shape in Play-doh or transform sand or Play-doh objects into one another. One teacher told two boys she was "going to hide the ball" of modeling clay, then covered it with a flat piece and pressed down. The boys said the ball was still there, but when she lifted the piece, the ball was "gone." This delighted them, and they copied her actions and discussed the idea that the ball was "in" the flat piece (Forman and Hill 1984, 31–32).

Seo and Ginsburg's study (2004) coded children's behaviors during free play of the pattern and shape category more frequently than the other six categories. About 47 percent of these behaviors involved recognizing, sorting, or naming shapes. However, children's capabilities exceed naming and sorting shapes. Ironically, geometry may be the richest mathematical topic in children's play, but it is the most neglected or oversimplified by adults who usually stop at naming a couple of basic shapes.

Block building is a prime example. Preschoolers use, at least intuitively, more sophisticated geometric concepts than most children experience throughout elementary school. For instance, they often produce symmetry in their play (Seo and Ginsburg 2004). One boy mentioned in the study put a double-unit block on the rug, two single-unit blocks on the double-unit block, and a triangle unit on the middle, thus composing a symmetrical structure. But, even teachers in middle school approach the topics of parallelism and perpendicularity with trepidation. They should not. Consider the study's account of a preschool boy making the bottom floor of a block building. He laid two long blocks down parallel to each other. Then he tried to bridge the two blocks with a shorter block. It did not span the space between the long blocks, so he moved an end of one of the long blocks to make it reach. However, before he tried the short block again, he carefully adjusted the other end of the long block. He seemed to understand that parallel lines are the same distance apart at all points. He then confidently placed the short block and followed quickly with the placement of many short blocks to create the floor of his building.

We learn a lot from this episode and others like it in Seo and Ginsberg's 2004 study. Many children intuitively use concepts of parallelism and perpendicularity just as this boy did. Such ideas have been called "theorems in action" (Vergnaud 1978). Other children were observed adjusting two cylinders so that the distance between them just equaled the length of a long block. They estimated how many more blocks they needed to finish a surface. They estimated that eight blocks were needed if each of four sizes of a square were covered with two blocks. We know many math teachers who would be thrilled if their students showed similar insight into geometry, measurement, and number.

Unfortunately, the same boy who by his actions seemed to understand that parallel lines are always the same distance apart may not understand these concepts when he arrives in middle school. If he is not helped to mathematize his theorems in his actions, they will not become theorems in his thought.

Children's play with blocks and other flat shapes that are designed to make pictures and designs, like puzzles, is also significant here. One observational study confirmed that the puzzle play of boys and girls was related to their mental-transformation ability (McGuinness and Morley 1991). However, controlling for the overall effects of parents' speech to children, socioeconomic status, and parents' spatial abilities, the use of spatial language by parents correlated only to the transformation skills of girls but not of boys. Parents' spatial language, such as "it's in the upper right-hand drawer," may be more important for girls (Cannon, Levine, and Huttenlocher 2007).

Research has also revealed a developmental progression of children's ability to compose geometric shapes, both in two-dimensional puzzle play and three-dimensional block building (Clements, Sarama, and Wilson 2001). Children at first are unable to combine shapes and can solve only the simplest puzzles, in which individual pieces only touch at their corners. Children gradually learn to see both individual pieces and a whole and learn that parts can make a whole and still remain parts. By about four years of age, most children can solve puzzles by trial and error and make pictures with shapes placed next to one another. With experience, they gradually learn to combine shapes to make larger shapes. They become increasingly intentional, building mental images of the shapes and their attributes, such as side length and angles. They can do this with physical blocks and computer shapes. Computer versions can give immediate feedback. Feedback can be even more helpful on computers, for example, highlighting shapes that do not fit and making those shapes transparent so children can see the outline puzzle "underneath" them. Further, children often talk more and explain more of what they are doing on computers than when using other materials (Clements and Sarama 2009). At higher levels, computers allow children to break apart and put together shapes in ways not possible with physical blocks. In our Building Blocks curriculum, we always have children play with both physical and computer manipulatives.

Computers can help facilitate play in other ways as well. The addition of a computer center does not disrupt ongoing play, but facilitates positive social interaction, cooperation, and helping (Rhee and Bhavnagri 1991; Rosengren et al. 1985). Computer activity is more effective in stimulating vocalization than toys and also evokes higher levels of social play (McCormick 1987). Finally, cooperative play at the computer matches the proportion of cooperative play in the block center (Anderson 2000). Cooperation in a computer center sometimes provides a context for initiating and sustaining interaction that can be transferred to play in other areas as well, especially for boys. For example, children may finish collaborating at the computer, then move to similar collaboration in playing with building blocks.

Constructive play is critical to high-quality preschools. There are hundreds of ideas for enhancing such play (see Forman and Hill 1984). A final example combines play, math, and physics. Changing the height of a ramp to see how it changes a toy car's speed or distance traveled lays a foundation for exploring and understanding functional relationships. The steeper the ramp (to a degree), the faster the car.

Symbolic Dramatic Play

Mathematics in constructive play is often enhanced when an element of the dramatic is added. In the right setting, sociodramatic play can be naturally mathematical. One suite of activities in the Building Blocks project (www.gse-buffalo.edu/org/buildingblocks) involves a Dinosaur Shop where children can purchase toys. Teachers and children put together a shop in the dramatic play area, and there the shopkeeper fills orders and asks the customer for money (we keep it simple—one dollar for each dinosaur toy).

In one classroom, Gabi was the shopkeeper. Tamika handed her a five card (five dots and the numeral 5) as her order. Gabi counted out five toy dinosaurs.

Teacher (just entering the area): How many did you buy?
Tamika: Five.

Teacher: How do you know?
Tamika: Because Gabi counted.

Tamika was still working on her counting skills and trusted Gabi's counting more than her own knowledge of five. The play context allowed her to develop her knowledge.

Janelle: I'm getting a big number. [She handed Gabi a two card and a five card.]
Gabi: I don't have that many.
Teacher: You could give Janelle two of one kind and five of another.

As Gabi counted out the two separate piles and put them in a basket, Janelle counted out dollars. She miscounted at first and gave her six dollars.

Gabi: You need seven dollars.

With the teacher's help, the sociodramatic play setting worked for children at three levels of mathematical thinking. Tamika learned to count and trust the results of her counting. Janelle explored place value. Janelle learned, and Gabi practiced, some arithmetic.

Play Supports Mathematical Thinking

We have seen the many ways different types of play bring forth and enhance mathematical thinking in children. Numerous studies show that if children play with objects before they are asked to solve problems with them, they are more successful and more creative (Bruner 1985).

For example, researchers gave three- to five-year-olds the task of retrieving an object with short sticks and connectors (Holton et al. 2001). One group was allowed to play with the sticks and connecting devices, one group was taught how the sticks could be connected, and one group was asked to tackle the task without prior play or learning. The play group and taught group performed similarly and achieved better results than the control group. Often, the play group solved the problem more quickly than the taught group. In other studies, the researchers assigned a final task of imagining several creative uses for some material. Again there were three groups: one designated free play; one, observe the experimenter; and one, solve specific, focused problems. Only the free-play group increased its creativity. Perhaps the play loosens the coupling between ends and means and allows for exploration of different combinations. In work on specific tasks, we hold the end steady and vary the means until we achieve our end. In play, we can also do the opposite.

Well, Almost Always—The Odd Occasion When Play Hurts Mathematical Understanding

Given the positive effects of play in so many situations, we find it hard to accept that, in a few particular settings, it can be harmful. Trouble sometimes occurs when we want children to represent something with something else—an important cognitive and mathematical goal. As an example, young children use simple maps and scale models to find objects in a room, and when we hide a miniature dog behind a small model couch, three-year-olds can usually find a larger stuffed dog hidden behind a full-sized couch (DeLoache 1987). But children six months younger can not. Moreover, younger children we allowed to play physically with the model were *less* successful! Playing with the model as a toy prevented the younger children from seeing it as a symbol of something else. However, children eventually develop the ability to see objects as both toys and symbols, and the value of play in the majority of situations remains clear.

Adults support math in play by providing a fertile environment and intervening appropriately. Play in perceptually oriented toddlers, for example, is enhanced by using realistic objects. And all children should also play with structured, open-ended materials. In both China and America, the use of LEGO and building blocks is strongly linked with mathematical activity in general and with pattern and shape in particular. However, U.S. preschools have many toys, some of which do not encourage mathematical activity. Chinese preschools have only a few play objects, and LEGO and building blocks are prominent (Ginsburg et al. 2003). In this case, "less is more."

Promoting Math in Everyday Play

Wise parents and educators take the time to observe children at play and intervene sensitively. When they fail to see new block constructions, they may share books illustrating different block structures or post pictures in the block center.

When they see children comparing sizes, they might offer different objects with which to measure, from cubes to string to rulers. A useful strategy is to ask if the social interaction and mathematical thinking is developing or stalled (Clements 2001). If it is developing, the adult observes but leaves the children alone, perhaps later discussing the experience with the whole class.

When mathematical thinking is stalled, the teacher can discuss and clarify the ideas. For example, one teacher heard two girls arguing about who had the bigger block tower. She observed them comparing the height of their towers to their bodies. Later, she asked them to explain to the class what they had done. The children recollected other times when they had compared how tall things were and brainstormed different ways they could measure height.

Adults—parents and teachers—should also check to ensure that they are being fair. Preschool teachers tend to spend more time with boys than girls in block, construction, and sand areas (Ebbeck 1984). Further, boys engage in spatial activities more than girls at home, both alone and with caretakers (Newcombe and Sanderson, 1993). We will return to such equity issues a little later, but for now let us simply say everyone, girls and boys, should have mathematical experiences. Adults should encourage girls as well as boys to build with blocks, for example.

In general, adults should schedule long periods of time for play and provide enriched environments and materials, including structured materials such as LEGO and building blocks that invite mathematical thinking. They should allow free use of classroom materials. They should be playful and respective in their interaction with children and give more attention, including longer periods of interaction with children engaged in dramatic play. All these enhance the benefits of dramatic play (Berkley 2000). Finally, adults should be alert to the mathematics that they might observe in children's play and be ready to alter materials and their interactions to give it encouragement, give it language, and give it their full appreciation.

Games with Rules

Turning to our last type of sociodramatic play, we note that games with rules could and should be a part of everyday play and also can be used intentionally to develop mathematical ideas. Most games can be introduced and modified to create opportunities to learn mathematical ideas, skills, and reasoning (Griffin 2004; Kamii and Housman 2000)—for example, games with number cards provide experiences with counting and comparison (Kamii and Housman, 2000). Card games such as Compare (War), Odd Card (Old Maid), and Go Fish can be used or adapted for learning mathematics and reasoning (Clements and Sarama 2004; Kamii and DeVries 1980). For example, children can fish for matching numbers or try to make a sum, such as five (e.g., a child with a two card would ask for a three card).

Games such as Memory (Concentration) use a different structure that encourages children to use memory strategies and gain experience with arrays (rows and columns). Children should be encouraged to declare when they found a match and how they knew where to find it. Computer versions can help motivate such play, whether the user plays alone or in pairs. Less desirable versions of the game allow children to win by random clicking until a match is found (and then automatically identified by the computer); a better design ensures children are recognizing the match.

Games such as tick-tac-toe also promote thinking about spatial relations and strategies, but they sometimes lead teachers to worry about early competition. But, first, competitive games such as this can help children learn to consider others' perspectives, i.e., I have to look at your *X*'s and into what square you might draw your next *X*. And, second, for many young children, winning a game merely means finishing, not beating others. Adults can encourage such sentiments and emphasize that the fun of playing games lies in cooperating and especially in playing smart.

Race or path games are similarly valuable. They usually include generating a number with number cubes (dice) or a spinner (roulette wheel) and moving the number of spaces indicated. This provides a different, complementary way of making sense of numbers, closely connected to measurement. According to some researchers, such connections build central conceptual structures for mathematics (Griffin and Case 1997). Other researchers have confirmed that such games develop children's number sense (Siegler and Ramani 2008).

Large group games are also valuable, such as Simon Says and other verbal games (Kamii and DeVries 1980). Games such as I Spy ("something with four sides the same length") or I'm Thinking of a Number (in which children seek clues of what numbers are smaller or larger than the one in mind) sharpen older children's knowledge of attributes and logical reasoning.

Good games promote more than concepts and skills—they encourage children to invent and test multiple strategies, to communicate, to negotiate rules and meanings, to cooperate, and to reason. Children should be encouraged to discuss rules, making up new ones when the group agrees. Adults should encourage children to discuss and evaluate their strategies, considering new approaches and solutions.

Mathematical Play

These and other examples bring us to the final, fascinating, and usually overlooked type of play: mathematical play. Here we do not mean play that involves mathematics—we've been talking about that throughout this article. We mean playing with mathematics itself.

Just after her fourth birthday, Abby was playing with three of the five identical toy train engines her father brought home. Passing by, her mother asked, "Where are the other trains?" Abby answered to herself, "Oh, I have five. Ummm." She pointed to each engine, "You are one, two, three. I'm missing four and five—two are missing!" She played with the trains for another minute. "No," she continued, "I have one, three, and five. I'm missing two and four. I gotta find them two" (Clements and Sarama 2009).

When Abby first figured out how many she was missing, she was using math in her play. But when she decided that she would renumber the three engines she had with her "one," "three," and "five" and the missing engines "two" and "four," she was playing with the notion that the assignment of numbers to a collection of objects is arbitrary. She was also counting not just objects, but counting words themselves. She counted the words "four, five" to confirm there were two missing and then figured that counting the renumbered counting words "two" and "four" *also* yielded the result of "two." She was playing with the idea that counting words themselves could be counted.

Research shows that the dynamic aspects of computers often engage children in mathematical play better than physical manipulatives or paper media (Steffe and Wiegel 1994). For example, two preschoolers were playing with the free exploration level of a set of activities called Party Time from the Building Blocks project. They could put out any number of items, and the computer counted and labeled the objects for them. "I have an idea!" said one girl, clearing off all the items and dragging placemats to every chair. "You have to put out cups for everybody. But *first* you have to tell me how many cups that'll be." Before her friend could start counting, she interrupted, "And everyone needs one cup for milk and one for juice!"

The girls worked hard together. They first used cups in the housekeeping center, and then replicated their solution by counting two times on each placemat on the screen. Their answer—initially nineteen using the physical cups—was not exact, but they were not upset to be corrected when they actually placed the cups and found they needed twenty. As they played with the software, these children also played with the mathematics of the situation.

Mathematics can be intrinsically interesting to children if they are building ideas while engaged in mathematical play (Steffe and Wiegel 1994). But to be interesting, instruction, both physical and computer materials, and verbal interactions must be of high quality. Ginsburg's writings contain many examples of such mathematical play. He also emphasizes that children like to play teacher, thus teaching and learning math at early ages.

One group of researchers describes the features of mathematical play, which can serve as a summary to this section: (a) it is a solution-centered activity with the solver in charge of the process; (b) it uses the solver's current knowledge; (c) it develops links between the solver's current schemes when the play occurs; (d) it reinforces, through the links developed, current knowledge; (e) it assists future problem-solving mathematical activity and enhances future access to knowledge; and (f) its behaviors and advantages occur irrespective of the solver's age (Holton et al. 2001).

Play as an Educational Approach in Schools

There is reason to support play in preschools, even for those concerned with the time devoted more directly toward learning. Several findings support the traditional emphasis on play and child-centered experiences. In one study, children made more progress overall and specifically in mathematics when they attended child-initiated, compared to strictly academically oriented, programs (Marcon 1992). There was some evidence that these children's test scores were higher at the end of elementary school (sixth, but not fifth, grade) (Marcon 2002).

This may be consistent with data from other countries. For example, Japanese preschool and kindergarten education places emphasis on social-emotional rather than academic goals. Preschoolers engage in free play most of the day. Parents deliberately interact with their children in mathematics, usually in real life, as in counting floors in an elevator. Few parents mention using workbooks (Murata 2004). Similarly, Flemish Belgium's preprimary education is concentrated more on overall development and less on specific content areas than education in the Netherlands (Torbeyns et al. 2002).

However, there are reasons for those interpreting this literature to be cautious. Marcon's studies have been criticized on methodological grounds (Lonigan 2003), and most of these studies are only correlational—there is no way to know what caused which effects. Further, exposure to mathematics instruction—not free-play time—explained a substantive portion of the greater gains of young Chinese, compared to U.S. children (Geary et al. 1996). Perhaps most troubling for an everyday or play-oriented approach to mathematics was that many programs with such a stated focus frequently showed negligible gains. One analysis showed that teaching math indirectly through everyday activities did not predict achievement gains, whereas group work did so (Klein et al. 2008). Probably, play alone—without adult guidance or interaction—did not have a strong influence on children's learning. Nevertheless, the importance of well-planned, free-choice play, appropriate to the ages of the children, should not be underestimated. Such play can build self-regulation skills, lay the foundation for mathematical understandings, and—if mathematized—contribute to mathematics learning. (For practical ideas for teaching, see Clements and Sarama 2005.)

Perhaps the most important caution we can offer is to ask: what is and is not an academic goal? Japanese preschool teachers, as we have said, distinguish themselves from elementary teachers in that they enhance social and emotional growth. What they mean is that instead of teaching numbers directly, they use materials such as card games, skipping ropes, scoreboards on which to write numerals, and so forth to induce quantitative thinking. (Hatano and Sakakibara 2004). Further, they enhance this activity by questioning the children or participating in their activities. They invite children who evidence a more advanced understanding to express their ideas in order to stimulate the thinking of other children (Hatano and Sakakibara 2004). Because Japanese culture highly values mathematical skills and concepts, such quantitative activities naturally occur frequently in children's lives. For example, during free play, a child took a few sheets of newspaper. Other children wanted some, and the teacher intervened and gave one sheet to each. She provided two rolls of tape. Some children started to create origami objects of their own, folding two edges into triangles. One child folded, saying, "Fold this into half. Fold this into half," making fourths (Hatano and Sakakibara 2004, 197). The teacher encouraged math learning by creating slightly more advanced paper objects. Children gathered around, and conversations developed about geometry and quantity. They began to make more complex objects of their own. They composed specific shapes, which they then discussed at length. Size and measure concepts threaded their conversations. Thus, these nonacademic teachers taught mathematics extensively. They arranged situations in which children could manipulate materials and discuss ideas; they offered increasingly challenging tasks; they helped children through modeling, participation, and provision of guidance; and they offered corrective or expanding feedback (Hatano and Sakakibara 2004). Thus, the very fact that such mathematical activity is ubiquitous in Japanese homes and schools indicates the degree of emphasis on preschool and kindergarten-aged children even compared to academic focus on mathematics in the elementary schools.

Observations also indicate that play can support mathematics learning if it stimulates learning and integrates the interests of children and educators (van Oers 1994). One observational study found that spontaneous use of mathematics in the play of children four- to seven-years-old was frequent enough that there were more teaching opportunities than a teacher could possibly notice, much less seize upon (van Oers 1996). Although the study used different mathematical categories than we have and observed just one dramatic play setting—a shoe store—it found children engaged in a wide variety of mathematical activities: classification, counting, one-to-one correspondence, measuring, estimating, solving number problems, simple arithmetic, quantitative concepts, number words, space-time, measurement, money, and seriation and conservation.

In another study, young children exposed to a play-based curriculum scored significantly higher than national norms for mathematics. However, the findings are equivocal, as the differences declined between ages five and seven to insignificance, and the children scored significantly lower than these norms in literacy. (Van Oers [2003] notes that the tests emphasize lower-level content.)

The Relationship between Preschool Mathematics and Literacy and Children's Everyday Play

Some adults believe that focusing more on mathematics and literacy will harm children's play, mainly by replacing free-play time with direct instruction. Research and practice indicate that such concerns are misplaced, for two reasons. First, as we have already described, mathematics instruction does not have to be only direct instruction. It can and should involve a variety of instructional approaches, including several different types of play. Second, and surprisingly, math and literacy instruction increase the quality of young children's play. Children in classrooms with stronger emphasis on literacy or math were more likely to be engaged in higher-quality free play. Those in classrooms with an emphasis on both literacy and math were more likely to be engaged in high-quality free play than those in classrooms with emphasis on only one or with no such emphasis (Aydogan et al. 2005).

Thus, high-quality instruction in math and high-quality free play need not compete for time in the classroom. Engaging in both makes each richer, and children benefit in every way.

Unfortunately, many adults believe that open-ended free play is good and lessons in math are not (Sarama 2002; Sarama and DiBiase 2004). They do not believe that preschoolers need specific math teaching (Clements and Sarama 2009). They do not realize that they are depriving their children not only the joy and fascination of mathematics, but of higher-quality free play as well.

Equity

All children must be provided opportunities to mathematize their informal experiences, abstracting, representing, and elaborating them mathematically, and using mathematical ideas and symbols to create models of their everyday activities. This includes the ability to generalize, to connect the mathematical ideas to different situations and use the ideas and processes adaptively. Research suggests a substantial and widening gap in mathematical achievement between children from higher-income and lower-income families, starting as early as three years of age (Sarama and Clements 2009). However, there are few, or no, differences between low- and middle-income children in the amount of mathematics they exhibit in their free play (Ginsburg et al. 2003; Seo and Ginsburg 2004). How can we make sense of this?

The apparent contradiction may have several explanations. Low-income children may not have the same informal opportunities at home (although there is only weak support for this hypothesis in Tudge and Doucet 2004), so perhaps they do not engage in mathematics in their play or other activities at home. These low-income children may engage in mathematics in their play in school but spend far less time in such play at home than higher-income children.

Another explanation is that low-income children may not have as many opportunities to reflect on and discuss their premathematical activities. There are large, meaningful differences in the sheer amount of language children from different income levels use (Hart and Risley 1995, 1999). Low-income children may engage in premathematical play but are not able to connect this activity to school mathematics because they lack opportunities to engage in the language and conversation necessary to bring implicit mathematical ideas to an explicit level of awareness. Research has found that a major difference between children from different socioeconomic backgrounds is not their ability to perform with physical objects but to solve problems verbally (Jordan, Huttenlocher, and Levine 1992) and to explain their thinking (Sophian 2002). Consider a child of four who, when given blocks and asked, "How much is ten and one more?" immediately counted the blocks, added one to ten, and answered, "Eleven." Five minutes later, when asked several times using the same wording, "How many is two and one more?" but without the blocks, she did not respond, and, asked again, said, "Fifteen" in a couldn't-care-less voice (Hughes 1981, 216–17). Research shows that higher-income children can solve problems both with and without objects (Jordan, Huttenlocher, and Levine 1992; Sarama and Clements 2009).

We believe the pattern of results suggest that, although low-income children have premathematical knowledge, they do lack important components of mathematical knowledge. Because they have been provided less support to learn, they lack the ability to connect their informal premathematical knowledge to school mathematics. As stated previously, we prefer to call most abilities learned in play "foundational abilities." Mathematization is requisite to basic mathematical ability. Adults must help children discuss and think about the mathematics they learn in their play. This is especially important for children from low-resource communities.

Conclusions

Young children engage in significant mathematical thinking and reasoning in their play, especially if they have sufficient knowledge about the toys or materials they are using, if the task is understandable and motivating, and if the context is familiar and comfortable (Alexander, White, and Daugherty 1997). Math can be seamlessly integrated into children's ongoing play and activities. But this usually requires a knowledgeable adult who creates a supportive environment and provides challenges, suggestions, tasks, and language. Such a supportive environment includes building and LEGO blocks, construction toys, card and board games, high-quality computer programs, and other materials. Such a knowledgeable adult helps children transform foundational play into mathematical knowledge and abilities. Children benefit from richer play experiences, preparation for learning later mathematics, and new ways to understand their world.

REFERENCES

Alexander, Patricia A., C. Stephen White, and Martha Daugherty. 1997. Analogical reasoning and early mathematics learning. In *Mathematical reasoning: Analogies, metaphors, and images*, ed. Lyn D. English, 117–48.

Anderson, Genan T. 2000. *An empirical comparison of the proportion of cooperative play of 4-year-old preschool children observed as they interact in four centers: Block, computer, housekeeping, and manipulative.*

Aydogan, Canan, Christy Plummer, Shin J. Kang, Carol Bilbrey, Dale C. Farran, and Mark W. Lipsey. 2005. An investigation of prekindergarten curricula: Influences on classroom characteristics and child engagement. Paper presented at the National Association for the Education of Young Children.

Berkley, Mira T. 2000. The teacher's role in facilitating sociodramatic play in early childhood classrooms: A study in Head Start classrooms. PhD diss., State University of New York at Buffalo.

Bruner, Jerome. 1985. On teaching thinking: An afterthought. In *Thinking and learning skills. Volume 2: Research and open questions,* ed. Susan F. Chipman, Judith W. Segal, and Robert Glaser, 597–608.

Cannon, Joanna L., Susan C. Levine, and Janellen Huttenlocher. 2007. Sex differences in the relation between early puzzle play and mental transformation skill. Paper presented at the biennial meeting of the Society for Research in Child Development.

Clements, Douglas H. 2001. Mathematics in the preschool. *Teaching Children Mathematics* 7:270–75.

Clements, Douglas H., and Julie Sarama. 2003. *DLM Math Software.*

Clements, Douglas H., and Julie Sarama. 2004. Building Blocks for early childhood mathematics. *Early Childhood Research Quarterly* 19:181–89.

Clements, Douglas H., and Julie Sarama. 2005. Math play: How young children approach math. *Early Childhood Today* 19:50–57.

Clements, Douglas H., and Julie Sarama. 2007. SRA real math building blocks pre-K.

Clements, Douglas H., and Julie Sarama. 2009. *Learning and teaching early math: The learning trajectories approach.*

Clements, Douglas H., Julie Sarama, and David C. Wilson. 2001. Composition of geometric figures. In *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education,* ed. Marja van den Heuvel-Panhuizen 2:273–80.

DeLoache, Judy S. 1987. Rapid change in the symbolic functioning of young children. *Science* 238:1556–57.

Ebbeck, Marjory. 1984. Equity for boys and girls: Some important issues. *Early Child Development and Care* 18:119–31.

Elkonin, Daniil B. 1978. *Psikhologija igry* [The psychology of play].

Forman, George E. 1982. A search for the origins of equivalence concepts through a microanalysis of block play. In *Action and thought: From sensorimotor schemes to symbolic operations,* ed. George E. Forman, 97–135.

Forman, George E., and Fleet Hill. 1984. *Constructive play: Applying Piaget in the preschool.* Rev. ed.

Geary, David C., C. Christine Bow-Thomas, Fan Liu, and Robert S. Siegler. 1996. Development of arithmetical competencies in Chinese and American children: Influence of age, language, and schooling. *Child Development* 67:2022–44.

Ginsburg, Herbert P., Lin Chia-Ling, Daniel Ness, and Kyoung-Hye Seo. 2003. Young American and Chinese children's everyday mathematical activity. *Mathematical Thinking and Learning* 5:235–58.

Ginsburg, Herbert P., and Robert L. Russell. 1981. Social class and racial influences on early mathematical thinking. *Monographs of the Society for Research in Child Development* 46:1–69.

Griffin, Sharon. 2004. Number worlds: A research-based mathematics program for young children. In *Engaging young children in mathematics: Standards for early childhood mathematics education,* ed. Douglas H. Clements, Julie Sarama, and Ann-Marie DiBiase, 325–42.

Griffin, Sharon, and Robbie Case. 1997. Re-thinking the primary school math curriculum: An approach based on cognitive science. *Issues in Education* 3:1–49.

Hart, Betty, and Todd R. Risley. 1995. *Meaningful differences in the everyday experience of young American children.*

Hart, Betty, and Todd R. Risley. 1999. *The social world of children learning to talk.*

Hatano, Giyoo, and Tomomi Sakakibara. 2004. Commentary: Toward a cognitive-sociocultural psychology of mathematical and analogical development. In *Mathematical and analogical reasoning of young learners,* ed. Lyn D. English, 187–200.

Holton, Derek, Afzal Ahmed, Honor Williams, and Christine Hill. 2001. On the importance of mathematical play. *International Journal of Mathematical Education in Science and Technology* 32:401–15.

Hughes, Martin. 1981. Can preschool children add and subtract? *Educational Psychology* 1:207–19.

Jordan, Nancy C., Laurie B. Hanich, and Heather Z. Uberti. 2003. Mathematical thinking and learning difficulties. In *The development of arithmetic concepts and skills: Constructing adaptive expertise,* ed. Arthur J. Baroody and Ann Dowker, 359–83.

Jordan, Nancy C., Janellen Huttenlocher, and Susan C. Levine. 1992. Differential calculation abilities in young children from middle- and low-income families. *Developmental Psychology* 28:644–53.

Kamii, Constance K. 1985. *Young children reinvent arithmetic: Implications of Piaget's theory.*

Kamii, Constance K., and Rheta DeVries. 1980. *Group games in early education: Implications of Piaget's theory.*

Kamii, Constance K., and Leslie B. Housman. 2000. *Young children reinvent arithmetic: Implications of Piaget's theory.* 2nd ed.

Kamii, Constance K., Yoko Miyakawa, and Yasuhiko Kato. 2004. The development of logico-mathematical knowledge in a block-building activity at ages 1–4. *Journal of Research in Childhood Education* 19:44–57.

Klein, Alice, Prentice Starkey, Julie Sarama, Douglas H. Clements, and Roopa Iyer. 2008. Effects of pre-kindergarten mathematics intervention: A randomized experiment. *Journal of Research on Educational Effectiveness* 1:155–78.

Kronholz, June. 2000. See Johnny jump! Hey isn't it math he's really doing? Yes or so hope advocates of standards that make lessons of preschooling. *Wall Street Journal,* May 16.

Lonigan, Christopher J. 2003. Comment on Marcon (ECRP, Vol. 4, No. 1, Spring 2002): Moving up the grades: Relationship between preschool model and later school success. *Early Childhood Research and Practice* 5:1. http://ecrp.uiuc.edu/v5n1/lonigan.html

Marcon, Rebecca A. 1992. Differential effects of three preschool models on inner-city 4-year-olds. *Early Childhood Research Quarterly* 7:517–30.

Marcon, Rebecca A. 2002. Moving up the grades: Relationship between preschool model and later school success. *Early Childhood Research and Practice* 4:1. http://ecrp.uiuc.edu/v4n1/marcon.html

McCormick, Linda. 1987. Comparison of the effects of a microcomputer activity and toy play on social and communication behaviors of young children. *Journal of the Division for Early Childhood* 11:195–205.

McGuinness, Diane, and Cindy Morley. 1991. Sex differences in the development of visuo-spatial ability in pre-school children. *Journal of Mental Imagery* 15:143–50.

Monighan-Nourot, Patricia, Barbara Scales, Judith L. Van Hoorn, and Millie Almy. 1987. *Looking at children's play: A bridge between theory and practice.*

Montie, Jeanne E., Zongping Xiang, and Lawrence J. Schweinhart. 2006. Preschool experience in 10 countries: Cognitive and language performance at age 7. *Early Childhood Research Quarterly* 21:313–31.

Murata, Aki. 2004. Paths to learning ten-structured understanding of teen sums: Addition solution methods of Japanese Grade 1 students. *Cognition and Instruction* 22:185–218.

Newcombe, Nora, and Hal L. Sanderson. 1993. The relation between preschoolers' everyday activities and spatial ability. Paper presented at the biennial meeting of the Society for Research in Child Development.

Oers, Bert van. 1994. Semiotic activity of young children in play: The construction and use of schematic representations. *European Early Childhood Education Research Journal* 2:19–33.

Oers, Bert van. 1996. Are you sure? Stimulating mathematical thinking during young children's play. *European Early Childhood Education Research Journal* 4:71–87.

Oers, Bert van. 2003. Learning resources in the context of play: Promoting effective learning in early childhood. *European Early Childhood Education Research Journal* 11:7–25.

Piaget, Jean. 1962, first published 1951. *Play, dreams, and imitation in childhood.*

Rhee, Mikyung C., and Navaz Bhavnagri. 1991. *4-year-old children's peer interactions when playing with a computer.*

Richardson, Kathy. 2004. Making sense. In *Engaging young children in mathematics: Standards for early childhood mathematics education,* ed. Douglas H. Clements, Julie Sarama, and Ann-Marie DiBiase, 321–24.

Rosengren, Karl S., Diane Gross, Anne F. Abrams, and Marion Perlmutter. 1985. An observational study of preschool children's computing activity. Paper presented at the Perspectives on the Young Child and the Computer Conference, University of Texas.

Sarama, Julie. 2002. Listening to teachers: Planning for professional development. *Teaching Children Mathematics* 9:36–39.

Sarama, Julie, and Douglas H. Clements. 2009. *Early childhood mathematics education research: Learning trajectories for young children.*

Sarama, Julie, and Ann-Marie DiBiase. 2004. The professional development challenge in preschool mathematics. In *Engaging young children in mathematics: Standards for early childhood mathematics education,* ed. Douglas H. Clements, Julie Sarama, and Ann-Marie DiBiase, 415–46.

Seo, Kyoung-Hye, and Herbert P. Ginsburg. 2004. What is developmentally appropriate in early childhood mathematics education? In *Engaging young children in mathematics: Standards for early childhood mathematics education,* ed. Douglas H. Clements, Julie Sarama, and Ann-Marie DiBiase, 91–104.

Siegler, Robert S., and Geetha B. Ramani. 2008. Playing linear numerical board games promotes low-income children's numerical development. *Developmental Science* 11:655–61.

Sophian, Catherine. 2002. Learning about what fits: Preschool children's reasoning about effects of object size. *Journal for Research in Mathematics Education* 33:290–302.

Steffe, Leslie P., and Heide G. Wiegel. 1994. Cognitive play and mathematical learning in computer microworlds. *Journal of Research in Childhood Education* 8:117–31.

Stiles, Joan, and Catherine Stern. 2001. Developmental change in spatial cognitive processing: Complexity effects and block construction performance in preschool children. *Journal of Cognition and Development* 2:157–87.

Stiles-Davis, Joan. 1988. Developmental change in young children's spatial grouping ability. *Developmental Psychology* 24:522–31.

Thirumurthy, Vidya. 2003. *Children's cognition of geometry and spatial thinking—A cultural process.* PhD diss., State University of New York at Buffalo.

Torbeyns, Joke, Wim van den Noortgate, Pol Ghesquière, Lieven Verschaffel, Bernadette A. M. van de Rijt, and Johannes E. H. van Luit. 2002. Development of early numeracy in 5- to 7-year-old children: A comparison between Flanders and The Netherlands. *Educational Research and Evaluation: An International Journal on Theory and Practice* 8:249–75.

Tudge, Jonathan R. H., and Fabienne Doucet. 2004. Early mathematical experiences: Observing young black and white children's everyday activities. *Early Childhood Research Quarterly* 19:21–39.

Vergnaud, Gerard. 1978. The acquisition of arithmetical concepts. In *Proceedings of the 2nd Conference of the International Group for the Psychology of Mathematics Education,* ed. Elmar Cohors-Fresenborg and Ipke Wachsmuth, 344–55.